Sciencia Acta Xaveriana
An International Science Journal
ISSN. 0976-1152

Volume 7
No. 1
pp. 19-27
March 2016

MAXIMUM INDEPENDENT SET COVER PEBBLING NUMBER OF COMPLETE GRAPHS AND PATHS

A.LOURDUSAMY[1], C. MUTHULAKSHMI @ SASIKALA[2], AND T. MATHIVANAN[3]

1,3 Department of Mathematics,
St.Xavier's College (Autonomous),
Palayamkottai-627 002, India.

2 Department of Mathematics,
Sri Paramakalyani College,
Alwarkurichi-627412, India.

1 lourdugnanam@hotmail.com
2 kalasasispkc@gmail.com
3 tahit_van_man@yahoo.com

Abstract. A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex. A graph is said to be cover pebbled if every vertex has a pebble on it after a sequence of pebbling moves. The maximum independent set cover pebbling number, ρ(G), of a graph G is the minimum number of pebbles that are placed on *V(G)* such that after a sequence of pebbling moves, the set of vertices with pebbles forms a maximum independent set of *G*, regardless of their initial configuration. In this paper, we determine the maximum independent set cover pebbling number of complete graph and paths.

Keywords: Graph pebbling, cover pebbling, maximum independent set, complete graph and path.

2010 Mathematics Subject Classification: 05C05, 05C69, 05C99

(Received : 23rd December 2014 ; Accepted : 20th March 2015)

1. INTRODUCTION

Graphs considered here are simple, finite, undirected, and connected. Given a graph *G*, distribute *k* pebbles on its vertices in some configuration. A pebbling move is defined by removing two pebbles from some vertex and placing one pebble on an adjacent vertex in which each move takes place along a path. The pebbling number [1], $\pi(G)$, of a graph G is the minimum number of pebbles that are placed on $V(G)$, such that after a sequence of pebbling moves, a pebble can be moved to any root vertex *v* in *G* regardless of the initial configuration. One can find the survey of graph pebbling in [3]. The cover pebbling number [2], $\gamma(G)$ of a graph *G* is defined as the minimum number of pebbles needed to place a pebble on every vertex of a graph *G* using a sequence of pebbling moves, regardless of the initial configuration. A set *S* of vertices in a graph *G* is said to be independent set (or an internally stable set) if no two vertices in the set *S* are adjacent. An independent set *S* is maximum if *G* has no independent set S' with $|S'| > |S|$. In [4], we have introduced the concept of maximum independent set cover pebbling number. The maximum independent set cover pebbling number, $\rho(G)$ of a graph *G*, is the minimum number of pebbles that are placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with pebbles forms a maximum independent set of *G*, regardless of their initial configuration. We have determined the maximum independent pebbling number of some families of graphs in [4, 5, 6]. In this paper, we determine the maximum independent set cover pebbling number ρ(G) for complete graphs and path graphs.

Notation 1.1. For any vertex a of *G*, $f(a)$ denotes the number of pebbles placed at the vertex *a*.

Notation 1.2. For $a, b \in V(G)$ and $ab \in E(G)$, $a \overset{m}{\rightarrow} b$ refers to moving *m* pebbles to *b* from *a*.

Notation 1.3. Throughout this paper we denote $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and P_n denotes the path $v_1, v_2, \ldots, v_n$.

2. MAXIMUM INDEPENDENT SET COVER PEBBLING NUMBER OF K_n AND P_n

Let us now find the maximum independent set cover pebbling number of complete graph K_n. Clearly $\rho(K_1) = 1$. We may get a feeling that, when we place a pebble on any of the vertices of K_n, $n \geq 2$, we get a maximum independent set cover pebbling. Since the maximum independent set cover pebbling number is the least possible integer, it should imply that we should get a maximum independent set cover pebbling for any number of pebbles greater than the maximum independent set cover pebbling number. In the case of the complete graph K_n, $(n \geq 2)$, the number of pebbles between 2 and n will not yield the property of maximum independent set cover pebbling.

Now we prove that $\rho(K_n) = n + 1$ for all $n \geq 2$.

Theorem 2.1. For K_n, $\rho(K_n) = n + 1$ $(n \geq 2)$.

Proof. Suppose a pebble is placed on each of the vertices $v_1, v_2, \ldots, v_n$ of K_n. Then we cannot cover a maximum independent set of K_n. Hence $\rho(K_n) > n$.

We use induction on n to show that $\rho(K_n) \leq n + 1$. First we prove that the theorem is true for $n = 2$.Consider the distribution of three pebbles on the vertices of K_2. If we place three pebbles on a single vertex, say v_1, then we are done. Otherwise by pigeonhole principle, there exists a vertex, say v_1, with exactly two pebbles. Then moving a pebble to v_2 from v_1 covers a maximum independent set of K_2. Thus $\rho(K_n) \leq 3$. Now consider the distribution of $n + 1$ pebbles on the vertices of K_n $(n > 2)$.By pigeonhole principle, there exists a vertex, say v_1, with at least two pebbles.

Case 1. $f(v_1) = 2$.

We move a pebble to the vertex v_2 from v_1. Clearly $f(V(K_n)-\{v_1\}) = n$, and the induced sub graph of $V(K_n)-\{v_1\}$ is K_{n-1}. Hence we are done by induction.

Case 2. $f(v_1) > 2$.

In this case, it is easy to see that there is a vertex, say v_i $(i \neq 1)$ with zero pebbles. Then $f(V(K_n)-\{v_i\}) = n + 1$ and the induced sub graph of $V(K_n)-\{v_i\}$ is K_{n-1}. Hence we are done by induction.Thus$\rho(K_n) \leq n+1$. ■

Let us now compute the maximum independent set cover pebbling number of a path P_n on n vertices. Since P_2 is isomorphic to K_2, $\rho(P_2) = 3$.

Theorem 2.2. For P_3, $\rho(P_3) = 6$.

Proof. Consider the following configuration: $f(v_2) = 5$ and $f(v_1) = f(v_3) = 0$. Then we cannot cover the maximum independent set of P_3. Thus $\rho(P_3) \geq 6$.

Now consider the distribution of six pebbles on the vertices of P_3.

Case 1. $1 \leq f(v_3) \leq 3$.

This implies that the path v_1v_2 contains at least three pebbles and hence we are done since ρ (P_2) = 3, except for the distribution $f(v_2)= 3$ and $f(v_3)= 3$. In this case consider the following sequence of pebbling moves: $v_3 \xrightarrow{1} v_2 \xrightarrow{2} v_1$ and hence we are done.

Case 2. $f(v_3) =0$.

We need at most four pebbles to put a pebble on v_3 from the vertices of $V(P_3)$-$\{v_3\}$. If we use three or four pebbles to pebble v_3 then we are done. Otherwise, $f(v_2) \geq 2$. If $f(v_2) = 2$ or $f(v_2) = 4$ then we are done by moving one or two pebbles to v_3 from v_2. If $f(v_2) = 6$ then we move a pebble to v_1 and two pebbles to v_3. For $f(v_2) =3$ or $f(v_2) =5$, we use two pebbles from v_2 to put a pebble at v_3. Then we can move all the pebbles from v_2 to v_1 using the pebbles at v_1 so that v_1 receives at least one pebble and hence we are done.

Case 3. $f(v_3) \geq 4$.

If $f(v_1) = 0$, then we apply Case 2. Let $f(v_1) \geq 1$. Then $f(v_2) \leq 1$. Suppose $f(v_2) =0$. Then we are done. If $f(v_2) =1$, then we move a pebble to v_2 from v_3 and then a pebble can be moved to v_1 from v_2 and hence we are done.

Thus ρ $(P_3) \leq 6$.

Theorem 2.3. For P_4, ρ $(P_4) = 6$.

Proof. Consider the following distribution: $f(v_1) = f(v_2) = 1$; $f(v_3)= 0$; $f(v_4)= 3$. Then we cannot cover a maximum independent set of P_4. Hence $\rho(P_4) > 5$. Now consider the distribution of six pebbles on the vertices of P_4. Let P_A be the subgraph induced by the vertices v_1, v_2 and let P_B be the subgraph induced by the vertices v_3, v_4. According to the distributions of six pebbles on the vertices of P_A and P_B, we consider the following two cases:

1. Both P_A and P_B contain exactly three pebbles.
2. Any one of P_A and P_B, say P_A, contains at most two pebbles.

Case 1. Both the paths P_A and P_B receive exactly three pebbles each.

Clearly we are done since $\rho(P_A) = \rho$ $(P_B) = \rho(P_2) = 3$, except for the distribution $f(v_2) = f(v_3)= 3$. Now we consider the following pebbling moves: $v_3 \xrightarrow{1} v_2 \xrightarrow{2} v_1$ and hence we are done.

Case 2. Assume that P_A contains at most two pebbles.

Subcase 2.1. Assume $f(P_A) = 0$.

Then $f(P_B)$ = 6 and we are done since $f(P_B \cup \{v_2\}) = 6$ and $P_B \cup \{v_2\}$ is isomorphic to P_3.

Subcase 2.2. Assume that P_A has a pebble on it.

Then P_B contains five pebbles. If $f(v_1) = 1$, then clearly we are done. So, assume that $f(v_2) = 1$. Then $f(P_B \cup \{v_2\}) = 6$ and $\rho\ (P_3) = 6$and hence we are done.

Subcase 2.3. Assume that P_A has two pebbles.

Then P_B contains four pebbles. If $f(v_1) = 2$ or $f(v_2) = 2$, then clearly we are done. Let $f(v_1) = 1$ and $f(v_2) = 1$. If $f(v_3) \geq 2$, then we are done. If $f(v_3) = 1$, then $f(v_4) = 3$. Consider the following pebbling moves: $v_4 \xrightarrow{1} v_3 \xrightarrow{1} v_2 \xrightarrow{1} v_1$ and we are done. If $f(v_3) = 0$, then $f(v_4) = 4$. Consider the following pebbling moves: $v_4 \xrightarrow{2} v_3 \xrightarrow{1} v_2 \xrightarrow{1} v_3$ and hence we are done.

Thus $\rho\ (P_4) \leq 6$. ■

Theorem 2.4. For P_5, $\rho\ (P_5) = 21$.

Proof. Consider the following configuration: $f(v_5) = 20$, $f(v) = 0$ for all $v \in V(P_5) - \{v_5\}$. Then we cannot cover the maximum independent set of P_5. Hence $\rho(P_5) > 20$. Let us consider the distribution of twenty one pebbles on the vertices of P_5 and different cases are discussed below. For that, let P_A be the subgraph induced by the vertices v_1 and v_2 and P_B be the subgraph induced by the vertices v_3, v_4 and v_5.

Case 1. $f(P_A) \leq 2$.

Then $f(P_B) \geq 19$. Let $f(P_A)=2$. If $f(v_1) = 2$ or $f(v_2) = 2$, then clearly we are done. So assume that $f(v_1) = 1$ *and* $f(v_2) = 1$. Using at most eight pebbles we can move a pebble to v_2 and then move a pebble to v_1. Hence the number of pebbles in v_2 is zero and we are done since $f(P_B) \geq 11$ and $\rho\ (P_B) = 6$. Let $f(P_A)=1$. Clearly we are done if $f(v_1) = 1$. Assume that $f(v_2) = 1$. Using at most eight pebbles from P_B, we can move a pebble to v_1, so that $f(v_2)$ becomes zero. Hence we are done, since $f(P_B) \geq 12$ and $\rho(P_B)=6$. Let $f(P_A)=0$. Using at most sixteen pebbles we can place a pebble on v_1. Then $f(P_B) \geq 5$. If $f(P_B) \geq 6$, then clearly we are done since $\rho(P_B)=6$. If $f(P_B)=5$ then $f(v_5) = 5$ and hence we are done.

Case 2. Assume $f(P_B) \leq 5$.

Then $f(P_A) \geq 16$. If $3 \leq f(P_B) \leq 5$, then clearly we are done. Let $f(P_B) \leq 2$. This implies that $f(P_A) \geq 19$. If $f(P_B)=2$ then also we are done. Assume that $f(P_B)=1$. Using at most sixteen pebbles from P_A, we can put one pebble each on v_3 and v_5 so that v_4 has zero pebbles on it. Then $f(P_A) \geq 4$ and we are done. If $f(P_B)=0$, then we can cover the maximum independent set of P_5 easily.

Case 3. Assume $f(P_A) \geq 3$ and $f(P_B) \geq 6$.

Clearly we are done except for the distribution $f(v_1) = 0$, $f(v_2) = 3$ and $f(P_B) = 18$. Using at most eight pebbles we can move a pebble to v_2 and then move two pebbles to v_1 from v_2. Hence we are done, since $\rho(P_3) = 6$ and $f(P_3) \geq 10$.

Thus $\rho(P_5) \leq 21$. ■

Theorem 2.5. For P_6, $\rho(P_6) = 21$.

Proof. Consider the following configuration: $f(v_6) = 20$, $f(v) = 0$ for all $V(G)-\{v_6\}$. Then we cannot cover a maximum independent set of P_6. Hence $\rho(P_6) > 20$.

Now consider the distribution of twenty one pebbles on the vertices of P_6. Let P_A be the subgraph induced by the vertices v_1 and v_2. Let P_B be the subgraph induced by the vertices v_3, v_4, v_5 and v_6. According to the distribution of these twenty one pebbles on the vertices of P_A and P_B, we find the following case:

Case 1. If $f(P_A) \leq 2$, then $f(P_B) \geq 19$.

Let $f(P_A) = 2$. If $f(v_1) = 2$ or $f(v_2) = 2$ then clearly we are done. So assume that $f(v_1) = 1$ and $f(v_2) = 1$. Then $f(P_B) = 19$. If $f(v_3) \geq 2$ then a pebble can be moved to v_2 from v_3 and then a pebble can be moved to v_3 from v_2. Then $f(P_B) \geq 18$ and we are done since $\rho(P_B) = \rho(P_4) = 6$. Assume $f(v_3) \leq 1$. If $f(v_3) = 1$, using at most eight pebbles from P_B we can move a pebble to v_3. And from v_3, a pebble can be moved to v_2 and then we can move a pebble to v_3 from v_2. Now $f(P_B) \geq 12$ and hence we are done since $\rho(P_B) = \rho(P_4) = 6$. If $f(v_3) = 0$, then using at most sixteen pebbles from P_B we can move a pebble to v_2. After moving a pebble to v_2 from P_B, if $f(P_B) \geq 6$ then we move a pebble to v_1 from v_2. If $3 \leq f(P_B) \leq 5$, then we move a pebble to v_3 from v_2. Clearly we are done, since v_6 is the only vertex contained *3* or *4* or *5* pebbles on it. Let $f(P_A) = 1$. If $f(v_1) = 1$ then we are done since $f(P_B) = 20$ and $\rho(P_B) = 6$. Similarly we are done if $f(v_2) = 1$. Let $f(P_A) = 0$. Then $f(P_B) = 21$. Clearly we are done since $f(P_B \cup \{v_2\}) = 21$ and $\rho(P_5) = 21$.

Case 2. If $f(P_B) \leq 5$, then $f(P_A) \geq 16$.

If $3 \leq f(P_B) \leq 5$, then using at most twelve pebbles, we move at most three pebbles to v_3 from P_A, so $f(P_B) \geq 6$ and we are done. Then $f(P_A) \geq 4$, hence we can pebble the maximum independent set of P_A, since $\rho(P_A)=3$. If $f(P_B)=2$, then $f(P_A)=19$. Using at most sixteen pebbles from P_A we can cover the maximum independent set of P_6. Assume that $f(P_B)=1$. Then $f(P_A)=20$. Suppose $f(v_6)=0$ and $f(v_i)=1$ for some $i=3, 4, 5$. Then we are done since $\rho(P_6-\{v_6\})=\rho(P_5)=21$. Suppose $f(v_6)=1$ and $f(v_i)=0$ for all $i=3, 4, 5$. Then using six pebbles we can cover maximum independent set of $P_6-\{v_5, v_6\}$ and we are done. If $f(P_B)=0$, then we are done since $\rho(P_6-\{v_6\})=\rho(P_5)=21$.

Case 3. $f(P_A)\geq 3$ and $f(P_B)\geq 6$.

Clearly we are done except for the distribution $f(v_1)=0$, $f(v_2)=3$ and $f(P_B)=18$. Since $f(P_B\cup\{v_2\})=21$ and $\rho(P_5)=21$, we are done in this distribution also.

Hence $\rho(P_6)\leq 21$. ■

Theorem 2.6. *For $P_n (n\geq 5)$,* $\rho(P_n)=\begin{cases}\frac{2^n-1}{3} & \text{if } n \text{ is even}\\ \frac{2^{n+1}-1}{3} & \text{if } n \text{ is odd}\end{cases}$

Proof. Consider the configuration where all pebbles are placed on the vertex v_1.

Clearly, we need at least $\begin{cases}\frac{2^n-1}{3} & \text{if } n \text{ is even}\\ \frac{2^{n+1}-1}{3} & \text{if } n \text{ is odd}\end{cases}$ pebbles to cover the maximum independent set $\begin{cases}\{v_1, v_3, v_5, \dots, v_{n-3}, v_{n-1}\} & \text{if } n \text{ is even}\\ \{v_1, v_3, v_5, \dots, v_{n-2}, v_n\} & \text{if } n \text{ is odd}\end{cases}$ of P_n from the vertex v_1.

Thus $\rho(P_n)\geq\begin{cases}\frac{2^n-1}{3} & \text{if } n \text{ is even}\\ \frac{2^{n+1}-1}{3} & \text{if } n \text{ is odd}\end{cases}$. Next we prove the upper bound by induction on n. The result is true for n = 5 and n = 6 from Theorem 2.4 and Theorem 2.5 respectively. Also note that $\rho(P_m)=\rho(P_{m-1})$ when m is even and for $n\geq 7$, $\rho(P_n)=\rho(P_{n-2})+\begin{cases}2^{n-2} & \text{if } n \text{ is even}\\ 2^{n-1} & \text{if } n \text{ is odd}\end{cases}$.

Now consider the distribution of $\rho(P_n)$ pebbles on the vertices of P_n.

Case 1. n is odd.

Let $f(v_n)=0$. If $f(v_{n-1})=0$, then we can pebble the vertex v_n by using at most 2^{n-1} pebbles. Then the path $P_{n-2}: v_1v_2v_3\dots v_{n-3}v_{n-2}$ contains at least $\rho(P_{n-2})$ pebbles and hence we are done. So, assume that $f(v_{n-1})\geq 1$. If $f(v_{n-1})=1$ or 3 then also we are done. If $f(v_{n-1})$ is even then we move a single pebble to v_n and then we move $\frac{f(v_{n-1})-2}{2}$ pebbles to v_{n-2}. Hence, we are done since $f(P_{n-2})+\frac{f(v_{n-1})-2}{2} \geq \rho\ (P_{n-2})$. If $f(v_{n-1}) \geq 5$ then consider the following sequence of pebbling moves: $v_{n-1} \xrightarrow{2} v_n \xrightarrow{1} v_{n-1} \xrightarrow{1} v_n$ and then we move $\frac{f(v_{n-1})-5}{2}$ to v_{n-2}. Here also, we are done since $f(P_{n-2}) + \frac{f(v_{n-1})-5}{2} \geq \rho\ (P_{n-2})$. So, we assume that $f(v_n) \geq 1$. In a similar way, we may assume that $f(v_1) \geq 1$. Consider the paths $P_A : v_1v_2...v_{n-2}$ and $P_B : v_3v_4...v_n$. Then, any one of the path contains at least $\rho(P_{n-2})$ pebbles. Without loss of generality, let P_A be the path. If $f(v_{n-1}) = 0$ then we are done easily. Similarly, we are done if $f(v_{n-1}) + f(v_n) \geq 2$ except the case $f(v_n) = 1$ and $f(v_{n-1}) = 1$. Now, we consider the case $f(v_n) = 1$ and $f(v_{n-1}) = 1$. For this case, P_A contains $\rho(P_n)$-2 pebbles on it. Using at most 2^{n-2} pebbles, we can move a pebble to v_{n-1} from the vertices of P_A and then we move a pebble to v_{n-2} from v_{n-1}. Thus, we are done since $\rho(P_n) - 2^{n-2} - 2 \geq \rho\ (P_{n-2})$.

Case 2. n is even.

Clearly, we are done if $f(v_n) = 0$ or $f(v_1) = 0$, since $\rho(P_n) = \rho(P_{n-1})$ when n is even. So, we assume that $f(v_n) \geq 1$ and $f(v_1) \geq 1$. Consider the paths $P_A : v_1v_2...v_{n-2}$ and $P_B : v_3v_4...v_n$. Then any one of the path contains at least $\rho(P_{n-2})$ pebbles. Without loss of generality, let P_A be the path. If $f(v_{n-1}) = 0$ then we are done easily. Similarly, we are done if $f(v_n) + f(v_{n-1}) \geq 2$ except the case $f(v_n) = 1$ and $f(v_{n-1}) = 1$. Finally, we consider the case $f(v_n) = 1$ and $f(v_{n-1}) = 1$. For this case, P_A contains $\rho\ (P_n)$-2 pebbles on it. Using at most 2^{n-2} pebbles from the vertices of P_A, we can move a pebble to v_{n-1}. If we use exactly 2^{n-2} pebbles to pebble v_{n-1} from P_A, then we move one pebble to v_{n-2} from v_{n-1} and hence we are done since the path $v_1v_2...v_{n-4}$ contains more than $\rho(P_{n-4})$ pebbles. If we use less than 2^{n-2} pebbles to pebble v_{n-1} from P_A, then we move one pebble to v_n from v_{n-1} and hence we are done since P_{n-2} contains at least $\rho(P_{n-2})$ pebbles.

Thus $\rho(P_n) \leq \begin{cases} \frac{2^n-1}{3} & \text{if} \quad \text{n is even} \\ \frac{2^{n+1}-1}{3} & \text{if} \quad \text{n is odd} \end{cases}$. ■

REFERENCES

[1]. F.R.K. Chung, Pebbling in hypercubes, *SIAM J. Disc Math,* **2** (1989), 467-472.

[2] B.Crul, T.Cundiff_, P.Feltman, G.H.Hurlbert, L.Pudwell, Z.Szanizlo, Z.Tuza, The cover pebbling number of graphs, *Discrete Math,* **296** (2005), 15-23.

[3] G.Hurlbert, A survey of graph pebbling, *Congressus Numerantium,* **139** (1999), 41-64.

[4] A. Lourdusamy, C. Muthulakshmi @ Sasikala, Maximum independent cover pebbling number of a star, *International Journal of Mathematical Archive*, **3(2)** (2012), 616-618.

[5] A. Lourdusamy, C. Muthulakshmi @ Sasikala and T. Mathivanan, Maximum independent set cover pebbling number of a Binary Tree, *Sciencia Acta Xaveriana*, **3(2)** (2012), 9-20.

[6] A. Lourdusamy, C. Muthulakshmi @ Sasikala and T. Mathivanan, Maximum independent set cover pebbling number of an *m*-ary tree, Accepted for publication (*Sciencia Acta Xaveriana*).